Riemann-Hilbert correspondence and Fukaya categories (based on a section from the paper in progress **Holomorphic Floer quantization**, joint with Maxim Kontsevich)

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Introduction (history and motivation)

1. Classically RH-correspondence relates algebraic vector bundles of rank r on a complex smooth projective curve X endowed with algebraic connections with RS at a finitely many points $S := \{x_1, ..., x_n\}$ (de Rham side) with representations of the fundamental group $\pi_1(X - S, x), x \notin S$ on $GL(r, \mathbb{C})$ (Betti side).
2. It was generalized by P. Deligne in 70's to higher-dimensional case as a theorem about equivalence of the category of algebraic vector bundles on a smooth algebraic variety X with flat connection and the category of local systems (i.e. locally-constant sheaves) on X.
3. At the beginning of 80's M. Kashiwara proved that the functor $M \mapsto \mathbf{R}Hom_{D_X-mod}(M, \mathcal{O}_X)$ gives the (derived) equivalence of the category of holonomic D-modules and the category of constructible sheaves. Under this equivalence the abelian category of holonomic D-modules with RS corresponds to the category of perverse sheaves. The functor associates the space of solutions (in derived sense) to a system of differential equations.

1. There is a microlocal version of the result of Kashiwara (Andronikoff, Waschkies, S. Gelfand-Macpherson-Vilonen) obtained around 2004. One replaces D-modules by modules over the sheaf of microlocal operators, and perverse sheaves by microlocal perverse sheaves (the notion introduced by the above authors). The definition of the latter relies on the notion of singular support $SS(F)$ of a constructible sheaf F introduced at that time by Kashiwara and Schapira (more algebraic treatment was proposed recently by T.Saito and Beilinson).

2. RH correspondence for irregular holonomic D-modules is more complicated, because the local description near singularities involves "Stokes data". In the case of curves and bundles with connections it is due to Deligne and Malgrange (80's). On the Betti side it involves filtrations of the local systems on circles about singularities. Higher-dimensional case is more recent and not as satisfactory as the one-dimensional case (T. Mochizuki, Sabbah, D'Agnolo, Kashiwara, Schapira and others contributed to its different aspects). All those results provide a correspondence between holonomic D-modules and constructible sheaves.

Aim of talk: to discuss several conjectures which relate RH-correspondence to (some versions of) the Fukaya category. The conjectures bring some new ideas to the old discussion about the relationship of Fukaya categories and deformation quantization (Bressler, Kapustin, Y.S., Tsygan, Witten,...).

From the point of view of the history recalled above, the relation of the RH-correspondence to Fukaya categories is not totally surprising. Indeed, by the work of Kontsevich, Nadler, Zaslow, Abouzaid and others some versions of the Fukaya category can be spelled out in the language of constructible sheaves. Representations of the fundamental group (Betti side of RH) form a subcategory of the category of constructible sheaves which describes $\mathcal{F}(T^*X)$ (Nadler-Zaslow). Replacing fundamental group by chains on based loops we recover the wrapped Fukaya category of T^*X (Abouzaid). We will need this story for a tubular neighborhood of a singular complex Lagrangian variety. The category of finite-dimensional modules over the corresponding wrapped Fukaya category, is (?) a subcategory of the corresponding Nadler-Zaslow version of the Fukaya category.

Few more motivating remarks

1. Although it seems that in the RH-correspondence the main role is played by constructible sheaves rather than Fukaya categories, this is not true, if we go beyond D-modules. For example, by the work of Ramis, Saloy and Zhang we know that in the RH-correspondence for D_q-modules on $\mathbb{C}^*$ (i.e. difference equations) one meets on the Betti side of RH coherent sheaves on an elliptic curve rather than constructible sheaves. The latter category admits a description in terms of an appropriate Fukaya category associated with a toric compactification of $(\mathbb{C}^*)^2$.
This remark also indicates the relationship of the RH-correspondence with deformation quantization (for which D- and D_q-modules are special cases).
2. From the point of view of the equivalence functor in RH, it will be useful to keep in mind the framework of family Floer homology: given a Lagrangian subvariety $L \subset M$ and a family of transversal Lagrangian submanifolds $F_x, x \in B$ we can construct a family of (graded in general) vector spaces $Hom_{\mathcal{F}(M)}(L, F_x)$ over the base B. This kind of objects should appear on the Betti side of the RH-correspondence.

Deformation quantization

Microdifferential operators give rise to a deformation quantization of T^*X. There are different versions of the latter, so we will need to specify the one we will work with. Let us recall relevant facts from deformation quantization.

Instead of the category of modules over the algebra of microdifferential operators on T^*X we now have the category of modules over an associative algebra $\mathcal{O}_{quant}(M)$ over $\mathbb{C}[[\hbar]]$, which deforms $\mathcal{O}(M)$, where $(M, \omega^{2,0})$ is a complex symplectic manifold (Kontsevich, Kashiwara, Schapira,...). More precisely,one has a sheaf of abelian categories. The notion of holonomic module is naturally defined by looking at the mod $\hbar$ support (which should be complex Lagrangian, but can be singular).

In 1998 Kontsevich propose the notion of semiformal deformation quantization in the case when M is an algebraic Poisson variety. We are going to use that version of deformation quantization.

Reminder on semiformal quantization

Let M be a complex Poisson (e.g. symplectic) smooth algebraic variety. We say that a normal crossing divisor compactification $\overline{M} \supset M$ is *quantizable* if:

a) $H^1(\overline{M}, \mathcal{O}_{\overline{M}}) = H^2(\overline{M}, \mathcal{O}_{\overline{M}}) = 0$,

b) the Poisson bivector field extends from M to $\overline{M}$,

c) it is tangent (in a precisely defined sense) to the divisor $D = \overline{M} - M$,

d) the class of the divisor D is ample in $Pic(\overline{M})$.

In the case when M is symplectic, the condition c) means that the symplectic form $\omega^{2,0}$ (considered as a rational differential form on $\overline{M}$) has non-trivial poles on D.

A quantizable compactification gives rise to the natural filtration of the algebra $\mathcal{O}(M)$ of regular functions with finite-dimensional terms of the filtration (by the orders of poles). The filtration makes $\mathcal{O}(M)$ into a filtered Lie algebra with respect to the Poisson bracket.

It was shown by Kontsevich that to any quantizable compactification $\overline{M}$ of M one can assign a canonical associative algebra $\mathcal{O}^{fin}_{quant} := \mathcal{O}^{fin}_{quant}(M, D)$ over $\mathbb{C}[[\hbar]]$ (called semicanonical quantization of M or of the Poisson algebra $\mathcal{O}(M)$) such that:
a) $\mathcal{O}^{fin}_{quant}/\hbar\mathcal{O}^{fin}_{quant} \simeq \mathcal{O}(M)$;
b) $\mathcal{O}^{fin}_{quant}$ has a compatible with the product exhaustive increasing filtration $\mathcal{O}^{fin}_{quant,\leq i}$ by finitely-generated $\mathbb{C}[[\hbar]]$-modules;
c) the filtration from b) admits a splitting over $\mathbb{C}[[\hbar]]$;
d) the Rees ring of the filtration $\mathcal{O}(M)_{\leq i}$ of $\mathcal{O}(M)$ induced mod $\hbar$ by the above filtration of $\mathcal{O}^{fin}_{quant}$ consists of finite-dimensional complex vector spaces.
Semiformal quantization gives rise to the formal deformation quantization: the $\hbar$-adic completion of $\mathcal{O}^{fin}_{quant}$ is a deformation quantization of $\mathcal{O}(M)$ in the formal sense. A semiformal quantization exists for affine M (e.g. for the cotangent bundle to a punctured curve).

In what follows we would like to think of $\hbar$ as of a complex number. For that reason we add an additional assumption which often holds in examples. We call semiformal quantization *algebraic* if the following *algebraicity assumption* holds:

The algebra $\mathcal{O}_{quant}^{fin}$ is defined over $\mathbb{C}[\hbar]$, and all properties a)-d) hold with $\mathbb{C}[[\hbar]]$ being replaced by $\mathbb{C}[\hbar]$.

For X affine and $M = T^*X$ an example is given by h-differential operators. In general one can think of a choice of compactification of M as a way to define a filtration on $\mathcal{O}(M)$ which can be lifted to its quantization. In this framework holonomic modules V can be defined via the power growth condition for $rk_{\mathbb{C}[\hbar]} gr(V)$. For $M = T^*X$ this guarantees that $dim(Supp(gr(V/\hbar V))) \leq n := dim\, X$ and together with Gabber's theorem implies that the support of a holonomic module is Lagrangian.

For the purposes of RH-correspondence for holonomic modules we will need partial compactifications of M, but of a special kind.

Log-extensions of complex symplectic manifolds

Let $(M, \omega^{2,0})$ be a complex algebraic symplectic manifold.

Definition

A log-extension of $(M, \omega^{2,0})$ is a smooth algebraic Poisson variety $M_{log} \supset M$ such that:
a) $M_{log} - M := D_{log}$ is a simple normal crossing divisor;
b) there exist local analytic coordinates $(x_1, ..., x_n, y_1, ..., y_n)$ near D_{log} such that D_{log} is given by the equation $\prod_{1 \le i \le k} x_i = 0$, and such that the holomorphic symplectic form on M can be written as

$$\omega^{2,0} = \sum_{1 \le i \le k} \frac{dx_i}{x_i} \wedge dy_i + \sum_{k+1 \le i \le n} dx_i \wedge dy_i,$$

for some $1 \le k \le n$.
In particular, the corresponding Poisson bracket vanishes on D_{log}.

Log-extension of an algebraic symplectic manifold is non-unique.

Log extensions for irregular D-modules on curves

Let X be a complex projective curve, $S \subset X$ a finite subset.
A singular term at a point $x_i \in S$ is a Puiseux polynomial in negative powers with respect to a local parameter: $c_\alpha(x) = \sum_{\lambda \in \mathbf{Q}_{\le 0}} c_{\alpha,\lambda}(x - x_i)^\lambda$.
Each polynomial c_α is assigned a multiplicity $m_\alpha \ge 1$. Then for a given choice of singular terms with multiplicities (+numerical conditions) there is a canonical choice of the log-extension M_{log} of the symplectic manifold $M = T^*X$.
First consider the fiberwise projectivization $\overline{T^*X}$. The standard holomorphic symplectic form $\omega^{2,0} = \omega^{2,0}_{T^*X}$ has poles of order 2 at the divisor at infinity $X_\infty \simeq X$. Then we construct a sequence of blow-ups $W_i = Bl_{p_i}(W_{i-1}), W_0 = \overline{T^*X}$ such that p_i is either a smooth point of a divisor in W_{i-1} where the pull-back of $\omega^{2,0}$ has pole of order ≥ 2 or p_i is the intersection of two divisors where the pull-back has poles of order ≥ 1. We start with blow-ups at points of $S_\infty \subset \overline{T^*X} - T^*X$ which is a copy of the set S. Finally we keep only those divisors where the pull-back of $\omega^{2,0}$ has poles of order 1. The union of T^*X and these divisors is our M_{log}.

Figure: blow-ups

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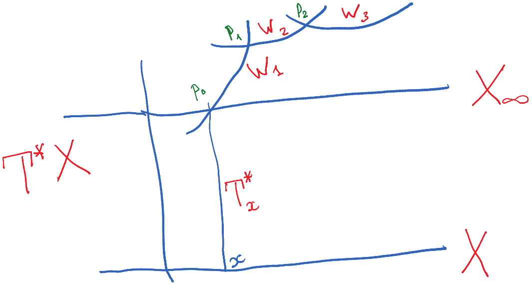

Introducing the parameter $\hbar$

Rescaling $\omega^{2,0} \mapsto \omega^{2,0}/\hbar$ we obtain a family of complex symplectic manifolds over $\hbar \in \mathbb{C}^*$. The maximal open symplectic leaf in the Poisson manifold M_{log} is naturally isomorphic to $M = T^*X$. Thus we obtain a family of *real* symplectic manifolds with respect to the symplectic form $\omega_\hbar = Re(\omega^{2,0}/\hbar)$. Let $L \subset M$ be a complex Lagrangian subvariety of M (i.e. curve) with compact closure $\overline{L} \subset M_{log}$ which intersects the divisor $D_{log} := M_{log} - M$ transversally in finitely many points. Its small neighborhood is Stein. Then we have a family over $\hbar \in \mathbb{C}^*$ of (finite-dimensional modules over the) partially wrapped Fukaya categories $\mathcal{W}F_{loc,\hbar}(L)$ (symplectic form is $\omega_\hbar$ and B-field is $Im(\omega^{2,0}/\hbar)$). They can be thought of as categories of finite-dimensional A_∞-modules over an A_∞-algebra $R(L)$ over $\mathbb{Z}$ (cf. Maxim's talk).
In general, for a quantizable complex symplectic M we can define $M_{log} := M \cup D_{log}$, where $D_{log} \subset D = \overline{M} - M$ is defined by the condition $ord_{D_{log}}\omega^{2,0} = 1$

Consider now the category of holonomic $\hbar$-D-modules on X such that for each $\hbar \in \mathbb{C}^*$ we obtain a D-module with the singular terms at given points $x_i, 1 \leq i \leq n$ equal to $c_\alpha/\hbar, \alpha \in J$ with multiplicity m_α, and the spectral curve isomorphic to a given L (the notion of spectral curve is well-defined for h-D-modules, as we can take a limit $\hbar \to 0$, but it is not well-defined for D-modules).
We denote the above $\mathbb{C}[\hbar]$-linear category by $Hol((c_\alpha/\hbar)_{\alpha\in J}, L)$. We also have a similar category $Hol((c_\alpha/\hbar)_{\alpha\in J})$ in which L is nor fixed. The latter can be also interpreted as the category of $\mathcal{O}^{fin}_{quant}(M)$-modules of finite type such that their support is a Lagrangian subvariety in $M = T^*X$ which has compact closure in M_{log} and finite transversal intersection with $D_{log} = M_{log} - M$. The former is a subcategory of the latter. Examples of objects in $Hol((c_\alpha/\hbar)_{\alpha\in J}, L)$ are $\hbar$-connections on a curve which modulo $\hbar$ are Higgs bundles with fixed spectral curve L. Our categories are linear over $\mathbb{C}[\hbar, \hbar^{-1}]$, so we can think of them as of families over $\mathbb{C}$.

One can define the (finite-dimensional modules over the partially wrapped) Fukaya category $\mathcal{W}F_{glob,\hbar}(M)$ of the open symplectic leaf $M \simeq T^*X$. Examples of objects are Lagrangian submanifolds of M with compact closure in M_{log}, intersecting the tubular neighborhood of each divisor of D_{log} at a circle and endowed with arbitrary local systems.

Conjecture

The categories $Hol((c_\alpha/\hbar))$ and $\mathcal{W}F_{glob,\hbar}(M)$ are equivalent for every $\hbar \in \mathbb{C}^*$.

The proof of this conjecture should follow from the comparison of both sides with categories of constructible sheaves. For holonomic D-modules we will have constructible sheaves F such that $SS(F)$ belongs to the union of X and all positive conormal bundles $T^{*,+}_{S^1_\alpha} \subset M$, where S^1_α is a closed curve $\theta \mapsto e^{Re(c_\alpha/\hbar)} \cdot e^{i\theta}$ understood as the image of a very small circle about a singular point (for bundles with connections follows from Deligne-Malgrange). Our Fukaya categories are invariants of the corresponding Legendrian links at infinity. Probably they are equivalent to those considered by Nadler, Shende, Treumann, Zaslow.

Conormal bundles

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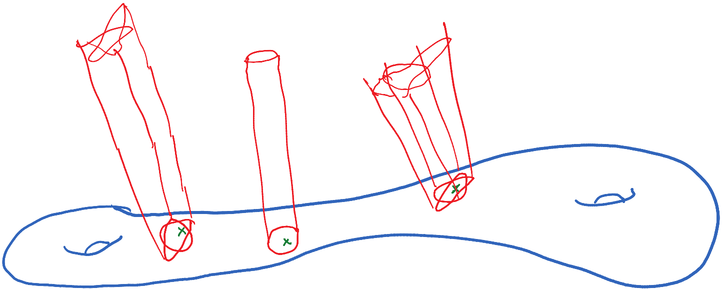

Our real analytic curves are projections of the curves in the space of contact elements of our Riemann surface. This gives a stratification of the latter for the corresponding constructible sheaves.
Fixing $\hbar \in \mathbb{C}^*$ we assign to an $\hbar$-D-module in our sense the usual D-module. In particular, to an object of $Hol((c_\alpha/\hbar)_{\alpha \in J}, L)$ we assign a D-module, which is (by the previous conjecture) corresponds to an object of the global Fukaya category of M.
On the other hand, we can compare local and global Fukaya categories with the aim to combine this comparison with the RH-correspondence for holonomic $\hbar$-D-modules. For the Fukaya categories we have the following *local-to-global* conjecture.

Conjecture

Assume that $\hbar$ does not belong to a Stokes ray $l_\gamma = \mathbb{R}_{>0} exp(\int_{\gamma \in H_2(M,L,\mathbb{Z})} \omega^{2,0})$. Then there is a fully faithful embedding $\mathcal{W}F_{loc,\hbar}(L) \to \mathcal{W}F_{glob,\hbar}(M)$.

Having an $\hbar$-D-module belonging to $Hol((c_\alpha/\hbar)_{\alpha\in J}, L)$ we can fix $\hbar \in \mathbb{C}^*$ and apply the above version of the RH-correspondence (with the global Fukaya category on the Betti side). The spectral curve L does not appear in such RH-correspondence, but it is natural to expect that as $\hbar \to 0$, the collection of the above RH-functors provides an equivalence of $Hol((c_\alpha/\hbar)_{\alpha\in J}, L)$ with $\mathcal{W}F_{loc,\hbar}(L)$, both considered as categories over the field $\mathbb{C}\{\hbar\}[\hbar^{-1}]$ of germs of meromorphic functions.
The story is in fact a bit more complicated on the Betti side, since the comparison of local and global Fukaya categories can be done inside of Stokes sectors only (i.e. sectors bounded by Stokes rays).

Consider the local system over $\mathbb{C}^*$ of the categories $\mathcal{W}F_{loc,\hbar}(L)$. It can be thought of as a category (linear over $\mathbb{C}\{\hbar\}[\hbar^{-1}]$) of finite-dimensional modules over the algebra $R(L)$. It receives "instanton corrections" from holomorphic discs in M with the boundary in L, when $\hbar$ hits a Stokes ray. The "instanton corrected" category is still an A_∞-category over $\mathbb{C}\{\hbar\}[\hbar^{-1}]$. Its objects can be described such as follows. In each Stokes sector they are holomorphic with respect to $\hbar \neq 0$ families of objects of $\mathcal{W}F_{loc,\hbar}(L)$ (equivalently, they are holomorphic families of finite-dimensional A_∞-modules over the A_∞-algebra $R(L)$).
For two adjacent Stokes sectors separated by the Stokes ray l the expansions at $\hbar = 0$ are connected by an isomorphism of $R(L)$ of the form

$$A_l = id + \sum_\alpha S_\alpha e^{-\frac{C_{\alpha,l}}{\hbar}},$$

where S_α is an integer matrix (in a certain basis), and $Re(-\frac{C_{\alpha,l}}{\hbar}) < 0$ on l, where $C_{\alpha,l}$ is the area of a holomorphic disc with the boundary on L.

Conjecture

The category $Hol((c_\alpha/\hbar)_{\alpha\in J}, L)$ *is equivalent to the above instanton corrected local Fukaya category over* $\mathbb{C}\{\hbar\}[\hbar^{-1}]$.

More precisely, the equivalence should hold over $\mathbb{C}[[\hbar]]$. At the same time, we expect that in many examples the series defining isomorphisms A_l converge as $|\hbar|$ is sufficiently small. For that reason the instanton corrected local Fukaya category is equivalent to $Hol((c_\alpha/\hbar)_{\alpha\in J}, L)$ over the field of meromorphic germs. Combining the above conjecture with the local-to-global one, we can assign a holonomic D-module to a complex Lagrangian variety L such that $H^1(L, \mathbb{Z}) = 0$ (there are no instanton corrections for all $\hbar$).

It would be interesting to compare the above story with the one for analytic continuations of the Laplace transforms (with respect to $1/\hbar$) of the WKB-expansions of flat sections of the corresponding $\hbar$-differential equations. The latter also receives exponentially small corrections ("Voros resurgence"). It would be nice to interpret them in "symplectic" terms.

For illustration purposes let us consider the RH-correspondence for $\hbar$-connections as $\hbar \to 0$. Then on the Betti side we have an analytic family of representation $\rho(\hbar)$ of the fundamental group of the curve. It does not have a limit as a representation of the fundamental group. If we fix a spectral curve L for the limiting Higgs bundle on the de Rham side, then on the Betti side we have the above data with categories in Stokes sectors and isomorphisms across Stokes rays (the latter are determined by periods $\hbar^{-1}\int_{\gamma\in H_1(L,\mathbb{Z})} pdq$). Thus we have a family of "usual" RH-equivalences for $\hbar \in \mathbb{C}^*$ and the equivalence of the category of $\hbar$-connections with given L with the above data derived from the local Fukaya category of L, as $\hbar \to 0$. If $L = graph(df_\alpha)$ (α counts sheets of the spectral curve), then the local Fukaya category gives $\hbar$-connections of the type $\sum_\alpha e^{f_\alpha/\hbar} \otimes (E_\alpha, \nabla_\alpha)$ visible on the Betti side as the family $Hom(L, T_x^*)$. The $\hbar$-connections derived from the local RH agree with those for fixed $\hbar$.

Transversals to SC

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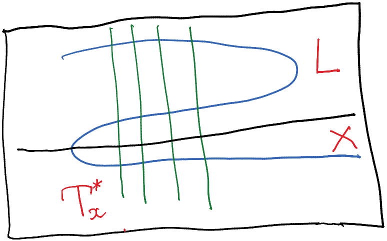

Spectral curve and family of transversal Lagrangians

1. This story resembles the one which appears in the discussion of SYZ mirror symmetry for the Fukaya category of the total space of Hitchin system discussed in our arXiv:1303.3253. In the loc.cit. the role of $\hbar$ was played by the twistor parameter ζ. We described there the corresponding log-extensions of M_{Betti} ("wild character variety") and the relation of all that to wall-crossing formulas and cluster varieties. In present story we also have wall-crossing formulas of Cecotti-Vafa type as $\hbar$ crosses a Stokes ray.
2. From the point of view of deformation quantization in any dimension, a choice of M_{log} and a complex Lagrangian variety $L \subset M$ such that $\overline{L}$ is compact in M_{log}, should give rise to a fully faithful functor from $\mathcal{W}F_{loc,\hbar}(L)$ to the category of holonomic modules (with respect to M_{log}) with support on L over the quantization of $\mathcal{O}(M)$ (at least for generic $\hbar$).

q-difference equations

Here we deal with modules over the algebra with generators $x_i, y_i, 1 \leq i \leq n$ and relations $x_i y_j = q^{\delta_{ij}} y_j x_i, 1 \leq i, j \leq n$ (x_i and y_i commute among themselves). We assume that $q = e^{\hbar}$ and $\hbar \notin i\mathbb{R}$, i.e. $|q| \neq 1$). The theory of D_q-modules makes sense on the torus $(\mathbb{C}^*)^n$. In many respects it is similar to the theory of algebraic differential operators on the affine space $\mathbb{C}^n$ (there exists an analog of Bernstein filtration, holonomic D_q-modules, direct and inverse image functors, b-function, etc.). It was developed in 90's by Sabbah. But the RH-correspondence is less understood. Even in the case $n = 1$ the analog of Stokes data was proposed relatively recently by Ramis, Saloy and Zhang (arXiv:0903.0853). Technically, they considered the case of vector bundles only.

Let $\mathcal{M}$ is a D_q-module on $\mathbb{C}^*$ (physicists meet such objects under the name of quantum spectral curves in $(\mathbb{C}^*)^2$). The corresponding sheaf of solutions V is defined similarly to the case of usual D-modules. It is a coherent sheaf on the the elliptic curve $E_q = \mathbb{C}^*/q^{\mathbb{Z}}$, since solutions of the equation $\psi(qx) = A(x)\psi(x)$ can be multiplied by "scalars" $\theta(qx) = \theta(x)$. Let $0 = V_{\lambda_0} \subset V_{\lambda_1} \subset ... \subset V_{\lambda_m} := V, \lambda_i \in \mathbf{Q} \cup \{\infty\}$ be a finite filtration of V by coherent subsheaves such that all quotients $F_{\lambda_i} := V_{\lambda_i}/V_{\lambda_{i-1}}$ are semistable and their slopes **increase** (infinity correspond to torsion sheaves). We call such a filtration anti-HN-filtration. Then the (slightly generalized) result of Ramis-Saloy-Zhang says:

Theorem

The category of holonomic D_q-modules on $\mathbb{C}^*$ is equivalent to the category of coherent sheaves on the elliptic curve E_q, which are endowed with two anti-HN filtrations labeled by $\mathbf{Q} \cup \{\infty\}$.

Two anti-HN filtrations describe Stokes data at $x = 0$ and $x = \infty$.

Relation to Fukaya categories

In the case of D_q-modules the role of T^*X is played by $M = (\mathbb{C}^*)^{2n}$. Then log-extensions correspond to toric compactifications, so that $\omega^{2,0} = \sum_{1 \leq i \leq n} \frac{dx_i}{x_i} \wedge \frac{dy_i}{y_i}$ has poles of degree 1 at the toric divisors. For $n = 1$ the global RH is roughly an equivalence between $\mathcal{W}F_{glob,\hbar}(M)$ and the category of algebraic D_q-modules with given Newton polygon. For the local RH-correspondence we choose a spectral curve L. The D_q-module arises from the Floer family homology construction. Namely, consider $Hom(L, (\mathbb{C}^*_x, E))$, where $\mathbb{C}^*_x, x \in \mathbb{C}^*$ is the transversal to L family of Lagrangians, and E is a rank 1 local system. This gives a 2-parameter family of vector spaces. The family is constant in log-coordinates along the 1-dimensional foliation induced by linear shifts along the directions x and E. The quotient of $(\mathbb{C}^*)^2$ by this foliation is our elliptic curve. Hence the Floer family homology construction gives a coherent sheaf on the elliptic curve. If it is a vector bundle, the Ramis-Saloy-Zhang result endows it with two anti-HN-filtrations.

It is not totally clear how to interpret anti-HN-filtrations in terms of Fukaya categories. There is a different framework in which anti-HN-filtrations appear naturally. Roughly, one considers semistable objects in the category $\mathcal{C} \times \mathcal{F}(\mathbb{R}^2)$, where $\mathcal{C}$ is any triangulated category with a stability structure, and $\mathcal{F}(\mathbb{R}^2)$ is the Fukaya category of $\mathbb{R}^2$.
The point is that if $\mathcal{C}$ "degenerates" in a certain sense, then semistable objects in the product category degenerate into graphs in $\mathbb{R}^2$ satisfying a kind of anti-HN-condition.
In our case $\mathcal{C} = D^b(Coh(\mathbb{C}^*/q^{\mathbb{Z}}))$.

Polygon

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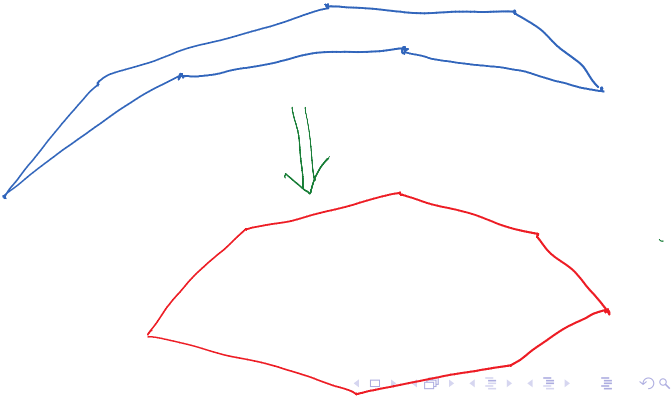

Final remarks

1. The above discussion of the RH-correspondence for holonomic D-modules on curves and holonomic D_q-modules on $\mathbb{C}^*$ can be generalized to higher dimensions in the following two cases: $T^*\mathbb{C}^n$ (case of D-modules) and $(\mathbb{C}^*)^{2n}$ (case of D_q-modules). After a choice of exact Lagrangian L with good behavior at infinity it gives e.g. in the case of $T^*(\mathbb{C}^n) = \mathbb{C}^{2n}$ an explicit description of the decomposition of the"sphere at infinity" S^{2n-1} into the union of domains generalizing the Stokes sectors. Each domain is endowed with a flag of vector spaces which are changed in a controlled way when we cross the boundary between two domains.' It is still conjectural. If true, it will give an explicit analog of the Deligne-Malgrange RH-correspondence and will make transparent the corresponding category of constructible sheaves. In the case of $(\mathbb{C}^*)^{2n}$ we should have a similar story with coherent sheaves on an abelian variety and many filtrations instead of two.

Then the inverse to RH-functor would guarantee an existence of the holonomic D-modules (or D_q-module) derived from the linear algebra data.

2. Our approach to the RH-correspondence gives rise (on the Betti side) to the data, which are essentially family Floer homology $V_\hbar$ depending holomorphically on $\hbar$ in Stokes sectors. They are subject to the wall-crossing formulas (when the value $Arg(\hbar)$ allows an existence of pseudo-holomorphic discs, i.e. "walls"). This story (which we call "Floer quantization") seems to be universal.
It appears in the study of exponential integrals $\int_C e^{f/\hbar} d\nu$ and their generalization to the case of closed 1-forms. Arising wall-crossing structures (notion introduced in our ArXiv: 1303.3253) appear in many applications, including analytic continuation of Chern-Simons theory, Khovanov homology, resurgence and many others.